

Homework 4

Math 121A, Fall 2025.

Due: Wednesday, Sep 24, 11:59 PM

Unless otherwise stated, all solutions require clear, well-written justification. You must explicitly credit any collaborators for each problem.

Problem 1. (*The many wonderful properties of orthogonal matrices*)

- (a) If O is orthogonal, then $O^T O = \mathbb{1}$. This means that $O^{-1} = O^T$.¹ So, $OO^{-1} = \mathbb{1}$ as well, and thus $\mathbb{1} = OO^T = (O^T)^T O^T$.
- (b) A matrix is invertible if there exists a matrix inverse O^{-1} such that $O^{-1}O = \mathbb{1}$. Observe that $O^{-1} = O^T$ is the inverse of O .

- (c) Observe:

$$\langle Ox, y \rangle = (Ox)^T y = x^T O^T y = \langle x, O^T y \rangle.$$

- (d) Observe:

$$\|Ox\|^2 = \langle Ox, Ox \rangle = \langle x, O^T Ox \rangle = \langle x, \mathbb{1}x \rangle = \langle x, x \rangle = \|x\|^2.$$

Take a square root, then we are done.

- (e) Consider the product $A = O^T O$. Its $(i, j)^{th}$ entry is exactly

$$(A)_{i,j} = v_i^T v_j = \langle v_i, v_j \rangle,$$

The identity matrix has 1s on the diagonal and zeros otherwise:

$$(\mathbb{1})_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Combining these two statements, a matrix O is orthogonal if and only if its columns form an orthonormal basis.

- (f) Taking the determinant of both sides of the equation $\mathbb{1} = O^T O$,

$$1 = \det(\mathbb{1}) = \det(O^T O) = \det(O^T) \det(O),$$

which since the determinant of the transpose is the determinant of the original matrix,

$$1 = (\det(O))^2.$$

Since O is a real matrix, the determinant is real, and so $\det(O)$ is either 1 or -1 .

¹Note that for matrices, every left inverse is a right inverse and vice versa.

(g) We may do this in several ways. Observe that the columns of O are orthonormal:

$$\left\langle \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \right\rangle = -\cos \theta \sin \theta + \sin \theta \cos \theta = 0,$$

and

$$\left\| \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \right\| = \cos^2 \theta + \sin^2 \theta = 1,$$

and likewise for the other column.

The determinant is $\det(R_\theta) = \cos^2 \theta + \sin^2(\theta) = 1$.

(h) This matrix is

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which has $\det(R) = -1$.

Problem 2. (*Google's most famous algorithm*)

(a) Since $M = M^T$, the spectral theorem for symmetric matrices guarantees that there is an orthogonal matrix O and a diagonal matrix D with real entries such that $M = ODO^T$.

(b) To diagonalize this matrix, we must compute its eigenvalues and eigenvectors. In this case it's not too hard to do this visually, but let's go through the calculation for thoroughness.

To find each eigenvalue λ , we find the roots of the characteristic polynomial:

$$\det(M - \lambda \mathbf{1}) = \left(\frac{1}{3} - \lambda\right)^2 - \left(\frac{2}{3}\right)^2.$$

We want to find roots of this, so set it equal to 0 and find

$$\frac{1}{3} - \lambda = \pm \frac{2}{3} \iff \lambda \in \{1, -1/3\}.$$

So, the eigenvalues are 1 and $-1/3$. Now, to find the corresponding eigenvectors, we must solve

$$Mv_\lambda = \lambda v_\lambda$$

for each one. In other words, we must compute

$$\text{rref}([M - \lambda \mathbf{1} | 0])$$

for each eigenvalue λ . We find that

$$\begin{aligned} \lambda = 1, v_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad \text{Check: } Mv_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = v_1 \\ \lambda = \frac{1}{3}, v_{1/3} &= \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \quad \text{Check: } Mv_{1/3} = \begin{pmatrix} 1/3 \\ -1/3 \end{pmatrix} = \frac{1}{3}v_{1/3}. \end{aligned}$$

S is a change-of-basis matrix from the eigenbasis of M to the standard basis, so its columns are eigenvectors of M :

$$S = (v_{-1/3} \quad v_1) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

(to verify this, check that $S \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v_{-1/3}$ and $S \begin{pmatrix} 0 \\ 1 \end{pmatrix} = v_1$).

So,

$$M = SDS^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1/3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

It is always a good idea to do a sanity check. We should verify this matrix decomposition by checking on the eigenbasis of M :

$$\begin{aligned} -\frac{1}{3}v_{-1/3} &= Mv_{-1/3} = SDS^{-1}v_{-1/3} = SD \begin{pmatrix} 1 \\ 0 \end{pmatrix} = S \begin{pmatrix} -1/3 \\ 0 \end{pmatrix} = -\frac{1}{3}v_{-1/3}. \\ v_1 &= Mv_1 = SDS^{-1}v_1 = SD \begin{pmatrix} 0 \\ 1 \end{pmatrix} = S \begin{pmatrix} 0 \\ 1 \end{pmatrix} = v_1. \end{aligned}$$

One more sanity check. We should see that this is in complete agreement with part (a).

The spectral theorem guarantees that the change-of-basis matrix can be chosen to be orthogonal. Notice that the columns of S are orthogonal, but not normalized—if we normalize them by letting $O = \frac{1}{\sqrt{2}}S$, we have the desired orthogonal change of basis:

$$M = ODO^T = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} -1/3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

(c) Observe that

$$M^2 = (SDS^{-1})(SDS^{-1}) = SD^2S^{-1} = S \begin{pmatrix} (-1/3)^2 & 0 \\ 0 & 1^2 \end{pmatrix} S^{-1} = \begin{pmatrix} 5/9 & 4/9 \\ 4/9 & 5/9 \end{pmatrix}.$$

Notice that we may in general calculate

$$M^n = SD^nS^{-1},$$

which is great because it is easy to calculate D^n for diagonal D but very hard to calculate M^n with raw matrix multiplication.

(d) We compute:

$$\begin{aligned} \lim_{n \rightarrow \infty} M^n &= \lim_{n \rightarrow \infty} SD^nS^{-1} = S \lim_{n \rightarrow \infty} D^nS^{-1} = S \lim_{n \rightarrow \infty} \begin{pmatrix} (-1/3)^n & 0 \\ 0 & 1^n \end{pmatrix} S^{-1} = S \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} S^{-1} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \end{aligned}$$

Problem 3. (*Our first quantum state, revisited*) The following matrix describes a famous single-qubit quantum state in superposition:

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

(a) Well, $\rho^\dagger = \rho$ just by looking at it. For the other one,

$$\rho^2 = \frac{1}{4} \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = \rho.$$

- (b) The eigenvectors are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. They are orthogonal, but not quite orthonormal, so we normalize to find $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Observe that

$$\rho v_1 = v_1$$

and

$$\rho v_2 = 0$$

which means that

$$[\rho]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

just like last week's homework.

- (c) It is much easier here to calculate

$$[\rho]_{\mathcal{B}}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = [\rho]_{\mathcal{B}}.$$

Problem 4. We say that a matrix A is *similar* to B if there is an invertible matrix C such that $A = CBC^{-1}$.

- (a) If A is similar to B , then there is an invertible matrix C such that $A = CBC^{-1}$. Multiplying on the left and the right, $C^{-1}AC = B$, so B is similar to A .
- (b) Let v be an eigenvector of A of eigenvalue λ , so

$$Av = \lambda v.$$

Notice that B also has an eigenvalue λ , corresponding to the eigenvector $C^{-1}v$:

$$B(C^{-1}v) = C^{-1}AC(C^{-1}v) = C^{-1}Av = C^{-1}\lambda v = \lambda(C^{-1}v).$$

- (c) The eigenvectors of B are given by the change of basis C^{-1} acting on the eigenvectors of A .

Problem 5. (*Function spaces, revisited*)

- (a) i. Positive definite:

$$\langle v, v \rangle = \int_0^1 v(x)v(x) dx = \int_0^1 v^2(x) dx \geq 0 \text{ since } v^2(x) \geq 0 \text{ for all } x \in [0, 1].$$

The definite part comes from the following proof idea: suppose towards contradiction that v is continuous and nonzero, say at $a \in [0, 1]$. By continuity there is a little region around a where v stays nonzero, say $v^2(x) > \delta > 0$ for all $x \in [a - \epsilon, a + \epsilon]$. In this case, $\int_0^1 v^2(x) dx \geq 2\epsilon\delta > 0$. This means that $\langle v, v \rangle = 0$ if and only if $v = 0$, the zero function.

- ii. Linearity:

$$\langle v, au + bw \rangle = \int_0^1 v(x)(au(x) + bw(x)) dx = a \int_0^1 v(x)u(x) dx + b \int_0^1 v(x)w(x) dx = a \langle v, u \rangle + b \langle v, w \rangle.$$

- iii. Symmetric:

$$\langle v, u \rangle = \int_0^1 v(x)u(x) dx = \int_0^1 u(x)v(x) dx = \langle u, v \rangle.$$

(b) This basis is not orthonormal. Let's do Gram-Schmidt.

$$v_1 = 1.$$

This has $\langle 1, 1 \rangle = \int_0^1 1 \, dx = 1$, so it is normalized. Now for the next one:

$$v_2^{(o)} = x - \langle x, 1 \rangle 1 = x - 1 \int_0^1 1x \, dx = x - 1\left(\frac{1}{2}\right).$$

This is not yet normalized, so

$$v_2 = v_2^{(o)} / \|v_2^{(o)}\|,$$

where

$$\|v_2^{(o)}\| = \sqrt{\int_0^1 (x - 1/2)^2 \, dx} = \frac{1}{\sqrt{12}}.$$

This means that

$$v_2 = \sqrt{12}(x - 1/2).$$

Finally, onto the last one.

$$\begin{aligned} v_3^{(o)} &= x^2 - \langle x^2, v_1 \rangle v_1 - \langle x^2, v_2 \rangle v_2 \\ &= x^2 - 1 \int_0^1 x^2 \, dx - 12(x - 1/2) \int_0^1 x^2(x - 1/2) \, dx \\ &= x^2 - x + 1/6, \end{aligned}$$

which we may normalize to

$$v_3 = \sqrt{180}(x^2 - x + 1/6).$$

Just in case, we should verify that these form an orthonormal basis:

$$\begin{aligned} \langle v_1, v_2 \rangle &= \int_0^1 1(\sqrt{12}(x - 1/2)) \, dx = 0 \\ \langle v_1, v_3 \rangle &= \int_0^1 1(\sqrt{180}(x^2 - x + 1/6)) \, dx = 0 \\ \langle v_2, v_3 \rangle &= \int_0^1 (\sqrt{12}(x - 1/2))(\sqrt{180}(x^2 - x + 1/6)) \, dx = 0, \end{aligned}$$

and they all have norm $\|v_1\| = \|v_2\| = \|v_3\| = 1$.

(c) From class, the projection onto V , P_V , can be calculated by

$$P_V \cos(2\pi x) = \langle \cos(2\pi x), v_1 \rangle v_1 + \langle \cos(2\pi x), v_2 \rangle v_2 + \langle \cos(2\pi x), v_3 \rangle v_3.$$

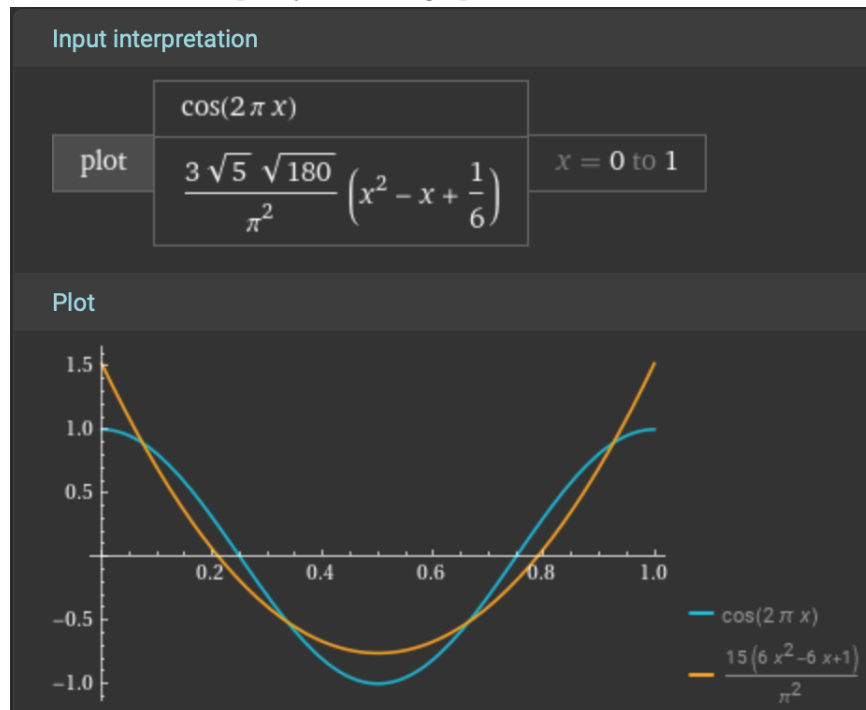
We can calculate each one separately:

$$\begin{aligned} \langle \cos(2\pi x), v_1 \rangle &= \int_0^1 \cos(2\pi x) \, dx = 0 \\ \langle \cos(2\pi x), v_2 \rangle &= \sqrt{12} \int_0^1 \cos(2\pi x)(x - 1/2) \, dx = 0 \\ \langle \cos(2\pi x), v_3 \rangle &= \sqrt{180} \int_0^1 \cos(2\pi x)(x^2 - x + 1/6) \, dx = \frac{3\sqrt{5}}{\pi^2}. \end{aligned}$$

So, altogether (if I didn't make an arithmetic mistake), we should have

$$P_V \cos(2\pi x) = \frac{3\sqrt{5}\sqrt{180}}{\pi^2} (x^2 - x + 1/6).$$

The formula is not pretty. But the graph sure is:



It took some work, but we calculated the degree 2 polynomial which *uniquely* minimizes the squared error, meaning it is the degree 2 polynomial for which

$$\|\cos(2\pi x) - p(x)\|^2 = \int_0^1 |\cos(2\pi x) - p(x)|^2 dx$$

is the smallest (I checked, the squared error is around 0.038). This is rather amazing! For reference, the squared error for the 2nd McLaurin polynomial is a whopping 68, not even in the same ballpark. This idea will return when we visit Fourier series in the last third of this course.

Problem 6. (*Spectra and matrix properties*) In the following let M be an $n \times n$ matrix.

- (a) First, assume M is not invertible. Then by the invertible matrix theorem, $\dim(\text{null}(M)) \geq 1$. Take a nonzero $v \in \text{null}(M)$: this means $Mv = 0$, so v is an eigenvector of eigenvalue 0. Now, assume M has 0 as an eigenvalue. Then there is a nonzero vector v with $Mv = 0$. Then $\dim(\text{null}(M)) \geq 1$, so the invertible matrix theorem implies M is not invertible.
- (b) By the spectral theorem, there is an orthonormal basis of eigenvectors $v_1, \dots, v_n$ of M :

$$Mv_i = \lambda_i v_i, \quad i = 1, \dots, n,$$

where λ_i are eigenvalues.

Now, if M is a contraction, that means

$$\|Mv_i\| < \|v_i\|,$$

but using that v_i is an eigenvector of eigenvalue λ_i , $\|Mv_i\| = \|\lambda_i v_i\| = |\lambda_i| \|v_i\|$, so the above becomes

$$|\lambda_i| \|v_i\| < \|v_i\| \iff |\lambda_i| < 1,$$

where we have just divided by $\|v_i\| > 0$ on both sides.

Now for the other direction. Any $w \in V$ may be expanded in the orthonormal eigenbasis:

$$w = w_1 v_1 + \cdots + w_n v_n.$$

Pick out the largest (absolute value of) the eigenvalues, $\lambda_{max} := \max_i |\lambda_i|$. By definition we have $|\lambda_i| \leq \lambda_{max}$ for all $i = 1, \dots, n$. Then using Pythagorean Theorem,

$$\|Mw\|^2 = \|w_1 \lambda_1 v_1 + \cdots + w_n \lambda_n v_n\|^2 = |w_1 \lambda_1|^2 + \cdots + |w_n \lambda_n|^2 \leq \lambda_{max}^2 \|w\|^2,$$

and then using the assumption that every λ_i has $|\lambda_i| < 1$, we have $\lambda_{max} < 1$ and the above becomes

$$\|Mw\|^2 \leq \lambda_{max}^2 \|w\|^2 < \|w\|^2.$$

By taking a square root of both sides we see that M is a contraction.

- (c) If v is an eigenvector of M of eigenvalue 1, then $Mv = v$. This just means that M does nothing to the vector v . In math, we say v is a *fixed point* of M .